The Jacobi-Gundelfinger-Frobenius-Iohvidov Rule and the Hasse Symbol

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This note, an addendum to the interesting book of Iohvidov [5] on Hankel and Toeplitz matrices, examines a small point in the inertia theory of real symmetric or Hermitian matrices, and its extension into the number theory of quadratic forms. For real quadratic forms (i.e., real symmetric matrices), this point stems from an identity of Jacobi and was substantially amplified by Frobenius, surely two of the most illustrious names in linear algebra, and concerns the determinantal criterion for specifying the inertia of a form. Our objective in this note is to show that the same insights extend without essential change into the number theoretic domain of a third giant name in linear algebra, Hasse, becoming a supplementary remark concerning the invariant usually called the Hasse symbol. The matrices to be investigated will be general symmetric matrices and symmetric matrices of the special classes usually identified with Hankel and Toeplitz.

Besides the illustrious names so far mentioned—Jacobi, Frobenius, Hasse, Hankel, Toeplitz (Sylvester could be included, since the law of inertia is attributed to him [but possibly Jacobi knew it earlier], and Hermite too, since Hermitian matrices were spoken of)—the point at issue sometimes also carries the name of a much less prominent mathematician, Gundelfinger. Indeed, we note that both Krein and Naimark's article [8] and Gantmacher's book [3] mention "Gundelfinger's rule" for real symmetric matrices. Where does Gundelfinger fit in? And what is the contribution of the modern mathematician Iohvidov?

Jacobi's identity was published posthumously [6], through Borchardt's initiative, and was a diagonalization from which it is evident that the inertia of a real symmetric matrix could be found by counting permanences or variations of signs in a nested chain of principal minors. The identity may be seen in Gantmacher's book, and in it the minors appear as denominator terms; thus none could be zero. Gundelfinger's contribution followed some years later, and was this: vanishing principal minors could be allowed in the

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count of sign changes, provided that two consecutive minors in the chain did not vanish. Frobenius's amplification came next, and consisted of two parts: first, the extension of the Jacobi rule to the case when two consecutive minors vanished, but not three; and second, a far-reaching extension to Hankel matrices, where he found a way of counting signs even though arbitrarily many consecutive minors vanished. Iohvidov's contribution, which was published in his book [5] and earlier papers, was that the Frobenius rule for giving signs to vanishing principal minors in Hankel matrices also applied to vanishing principal minors in Toeplitz matrices. In the process of obtaining this extension, Iohvidov added substantially to the general theory of both Hankel and Toeplitz matrices.

Following historical order, therefore, we speak of the Jacobi-Gundelfinger-Frobenius-Iohvidov rule, and we wish to study it in the context of the Hasse symbol. The reason for doing so is that the exact analogue of Gundelfinger's rule occurs in the number theory of quadratic forms, though without mention of Gundelfinger's name. The number theory version of the rule appears in the well-known Carus monograph [7] of Burton W. Jones, and appears also in the unpublished notes of Gordon Pall [10]. Neither of these sources examines extensions of the type considered by Frobenius.

After the many historically significant names that have been mentioned, with great humility the present author mentions his *very* small contribution, doing so only in order to complete the historical record. Although Frobenius was able to permit arbitrarily many consecutive vanishing minors in Hankel matrices, he could only allow two consecutive vanishing principal minors for general symmetric matrices, observing that no theorem existed when three consecutive principal minors vanished. It turns out, though, that there is a tiny theorem when three consecutive minors vanish, but no theorem at all (for general symmetric matrices) if four consecutive principal minors are allowed to vanish. See [11].

The present note is this author's second contribution to this class of questions, and its point is: Everything that Jacobi, Gundelfinger, Frobenius, and Iohvidov did relative to these rules of signs carries over to the analogue of inertia in number theory, namely, to the computation of the Hasse symbol. This includes the very small contribution of the last paragraph. The number theory analogues of the theorems of Frobenius and Iohvidov seem to be new results, although they probably will not surprise any expert from quadratic form theory, since the underlying reason for the truth of these theorems is the same in both the real and number theoretic cases.

We now give a more detailed discussion. Let S be an $n \times n$ nonsingular real symmetric matrix. The Sylvester law of inertia asserts that any diagonalization $S \to MSM^{T}$ of S under a real congruence transformation leads to the same number of positive diagonal elements. Here M is real and nonsingular, and $^{\mathsf{T}}$ denotes transposition. This constant number of positive diagonal elements when S is diagonalized is sometimes called the positive index of S. When S is viewed as a Gram matrix associated with a nondegenerate quadratic space, the positive index is the maximal dimension of a positive definite subspace. The Jacobi rule, as supplemented by Gundelfinger, characterizes the positive index solely in terms of the elements of S itself. Select a nested chain

$$S_1 \subset S_2 \subset \cdots \subset S_n$$

of principal submatrices of S, where S_i is $i \times i$ for $1 \leqslant i \leqslant n$. Set $D_i = \det S_i$, and consider the sequence

$$D_0 = 1, D_1, D_2, \ldots, D_n. \tag{1}$$

The Jacobi rule is that the positive index equals the number of permanences of sign in the sequence (1), provided that no term of (1) is zero. Gundelfinger's supplement is that if zeros occur, but are isolated, the zeros may be given any sign (positive or negative). Without this supplement, the Jacobi rule is inadequate, since matrices exist in which every nested chain of minors contains some zeros. With the supplement, it is adequate in principle, since it can be proved that a nonsingular matrix will always contain at least one nested chain of minors in which the zeros are isolated.

However, an arbitrarily selected nested chain of principal minors often will contain nonisolated zeros. According to Frobenius [2], though, when two consecutive zeros are present, but not three, a fixed rule exists for replacing the zeros with nonzero values in a way that restores the permanences of sign assertion; see (2) below. No extension to three consecutive zeros is possible in general, but is possible when three consecutive vanishing terms in (1) are proceeded and followed by terms of the same sign. This first appeared in [11], and (2) is the rule. No extension to four consecutive zeros is possible. All these statements hold for Hermitian as well as for real symmetric matrices, congruence being changed to conjunctivity.

A Hankel matrix is a matrix constant on each diagonal perpendicular to the main diagonal. Such a matrix necessarily is symmetric. If S is a real, nonsingular Hankel matrix, and if (1) is the nested chain of *leading* principal minors, with some zero terms possibly present, Frobenius found that prescribing nonzero values for the zero terms in the following way restores the Jacobi rule:

If $D_{i-1} \neq 0$, but $D_i = D_{i+1} = \cdots = D_{j-1} = 0$, and $D_j \neq 0$, change $D_{i-1+t} = 0$ to $D_{i-1}(-1)^{t(t-1)/2}$ for $t = 1, \ldots, j-i$.

(2)

(This prescription is also given in Chapter 10 of Gantmacher's book.) Iohvidov found that the same prescription (2) restores Jacobi's permanences of sign rule for Hermitian, nonsingular Toeplitz matrices. (A matrix is Toeplitz if it is constant along each diagonal parallel to the main diagonal.)

[When the vanishing minors are isolated, but only then, one may change zeros to nonzeros in any way, not necessarily (2), as Gundelfinger had already observed.]

In the number theory of quadratic forms, a basic invariant is the Hasse symbol. Various definitions of it appear in the literature, differing at most by a constant factor. Slightly modifying the definition in Pall's notes [10] and Jones's book [7] to make it agree with O'Meara [9], we specify the Hasse symbol $c(S)$ of a symmetric matrix S over a local field of number theory by

$$c(S)=(-1,D_n)\prod_{i=1}^{n-1}(D_i,-D_{i+1}),\tag{3}$$

where (,) is the Hilbert symbol: $(\alpha,\beta)=1$ if $\alpha x^2+\beta y^2=1$ has a solution x, y in the local field; otherwise $(\alpha,\beta)=-1$. This definition of $c(S)$ assumes that no D_i is zero. However, if there are some zero D_i, but the zeros are isolated, both Jones and Pall show that the Gundelfinger supplement to Jacobi's rule also works in the Hasse symbol computation: replace a zero D_i by an arbitrary nonzero value. And now we ask: do the various other extensions of the Jacobi law described above apply to the Hasse symbol computation? The answer is yes, and the reason is the same in every case: the underlying quadratic space splits off hyperbolic planes. The remainer of this paper outlines the proof of this "yes."

We henceforth assume that nonsingular symmetric matrix S has entries in a local field of number theory; see [9] for the definition of a local field. In particular, S may have entries in any p-adic closure of the rational number field Q, possibly in Q itself. (Alternatively, see [1].)

Our theorem may now be stated as follows:

THEOREM. *The Frobenius prescription* (2) *for substituting nonzero values for vanishing principal minors in nonsingular symmetric matrix S is valid in the Hasse symbol formula* (3) *for S in these cases:*

(i) *when S is an arbitrary symmetric matrix, whenever*

(a) *a zero minor is isolated;*
(b) *two consecutive minors D_i, D_{i+1} vanish, but not three;*
(c) *three consecutive minors D_i, D_{i+1}, D_{i+2} vanish, but not four, provided D_{i-1} and D_{i+3} belong to the same local square class.*

(ii) *When S is a Hankel or a Toeplitz matrix and the principal minors are in leading position, always.*

Proof. We write $S \cong T$ when matrices S and T are congruent by a nonsingular matrix with entries in the base field. No generality will be lost by taking the nested chain of minors to be in leading position, this being in fact required in the Hankel case.

Assume that i is maximal with $D_{i-1} \neq 0$, $D_i = 0$. Then S_{i-1} is nonsingular and $S \cong S_{i-1} \dotplus S'$ under a triangular congruence preserving leading principal subdeterminants up to squares; $\dotplus$ is direct sum. Because $D_i = 0$, the leading element of S' is zero. In (i)(a) the leading 2×2 submatrix of S' is nonsingular, hence after a further congruence,

$$S \cong S_{i-1} \dotplus \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} \dotplus S'' \cong S_{i-1} \dotplus \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \dotplus S'',$$

where x is an arbitrary nonzero value, the 2×2 congruence in the last step being nontriangular. (Leading minors of S extending into S'' are preserved up to squares.) This congruent form of S may be used to compute the Hasse symbol, since the symbol is invariant under congruence, and in this form we have $D_i = xD_{i-1}$ with x arbitrary nonzero. This settles case (i)(a).

In case (i)(b), the leading 2×2 minor in S' vanishes but the leading 3×3 does not, so that the top row of S' has the form $(0, 0, a, \dots)$ with $a \neq 0$. By a triangular congruence,

$$S' \cong \begin{bmatrix} 0 & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & 0 \end{bmatrix} \dotplus S'',$$

with $b \neq 0$, and by a nontriangular 3×3 congruence, $S' \cong \operatorname{diag}(1, -1, b) \dotplus S''$. Using the congruent form of S to compute the Hasse symbol, we have $D_i = D_{i-1}$, $D_{i+1} = -D_i$, in accord with the Frobenius rule.

In case (i)(c), several possibilities have to be considered. We have

$$S' = \begin{bmatrix} 0 & 0 & a & c & \cdot \\ 0 & b & d & e & \cdot \\ a & d & \cdot & \cdot & \cdot \\ c & e & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

with $ab = 0$. If $a \neq 0$, then $b = 0$ and a triangular congruence permits us to take $d = 0$, $e \neq 0$. After a further triangular congruence,

$$S' \cong \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \\ a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \end{bmatrix} \dotplus S'' \cong \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \dotplus S''.$$

A nontriangular 4×4 congruence permits us to pass S' into the form diag $(1,-1,1,-1)\dotplus S''$. For the S now at hand, $D_i = D_{i-1}$, $D_{i+1} = -D_{i-1}$, $D_{i+2} = -D_i$, and the Frobenius rule holds. If $b\neq 0$, then $a=0$, $c\neq 0$, and a triangular congruence passes S' to

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & f & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \dotplus S''.$$

Because D_{i-1} and D_{i+3} are in the same square class, $f=-bx^2$ for some nonzero x from the base field. A nontriangular 4×4 congruence moves us to $\operatorname{diag}(1,-1,b,-b)\dotplus S''\cong \operatorname{diag}(1,-1,1,-1)\dotplus S''$, from which the Frobenius rule is clear. If $a=b=0$, then $c\neq 0$, $d\neq 0$, and a triangular congruence converts S to

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \dotplus S'' \cong \operatorname{diag}(1,-1,1,-1)\dotplus S'',$$

the last 4×4 congruence being nontriangular. Again the Frobenius prescription holds for the form at hand.

The same reasoning may now be applied to the last set of vanishing leading minors in S_{i-1}. By induction, all parts of (i) are proved.

Now let S be a Hankel matrix, and as in [5], label the rows $0,1,\ldots,n-1$, instead of $1,2,\ldots,n$. Let i be maximal such that $D_{i-1}\neq 0$, $D_i = D_{i+1} = \cdots = D_{j-1} = 0$, $D_j \neq 0$. We invoke results from [5, Chapter 2]. We have

$$S_j = \begin{bmatrix} S_{i-1} & \cdot \\ \cdot & \cdot \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \cdot \end{bmatrix},$$

where the first matrix on the right is a singular prolongation of S_i and the second contains the "broken diagonal." (See Iohvidov's book, Chapter 2, for these technical terms.) A triangular congruence passes S_j to

$$S_{i-1} \dotplus \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & b \\ 0 & 0 & \cdot & \cdot & \cdot & b & 0 \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ 0 & b & \cdot & \cdot & \cdot & 0 & 0 \\ b & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}, \qquad b\neq 0.$$

A nontriangular congruence on the second block here converts it to diag $(1,-1,1,-1,\ldots,\pm 1,b)$, with the b possibly absent. From this we read the validity of the Frobenius rule. Since S_{i-1} is still Hankel (in fact, unchanged), we may repeat the argument and proceed by descending induction.

Finally, let S be Toeplitz and symmetric. We now invoke results from Chapter 3 of [5]. With notation as in the Hankel case, $j-i$ must be even, and

$$S_j = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & S_{i-1} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \dotplus \begin{bmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ B^{\mathrm{T}} & 0 & 0 \end{bmatrix},$$

where the first matrix on the right is a singular prolongation of S_i, and in the second B is triangular with a nonzero principal diagonal (the "broken diagonal"). Then $S_j \cong S_{i-1} \dotplus \operatorname{diag}(1,-1,1,-1,\ldots,1,-1)$, and the result is at hand. This ends the proof. ∎

REMARK on the singular case: If symmetric S is singular, of rank ρ, say, it is known [10] that a Hasse symbol for S may be obtained by computing the Hasse symbol for any nonsingular $\rho\times\rho$ principal submatrix. When S is Hankel with $D_{r-1}\neq 0$, $D_r = D_{r+1} = \cdots = D_{n-1} = 0$, one may (following Iohvidov and Frobenius) use the $\rho\times\rho$ principal submatrix formed from the first r and last $n-r$ rows (and columns). The Frobenius rule applies to vanishing leading subdeterminants in this submatrix of S. When S is Toeplitz, one may use the principal submatrix taken from the first $\frac{1}{2}(\rho-r)$ rows, the last $\frac{1}{2}(\rho-r)$ rows, and any set of r consecutive rows (and the same columns) between. We omit the details, as they follow quite directly using Chapters 2 and 3 of [5].

(We mention, in passing, that an English translation of [5] exists in which certain refinements to several proofs have been made. See [12].)

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